JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 18, No. 3, May–June 1995

Optimization of Low-Thrust Interplanetary Trajectories Using Collocation and Nonlinear Programming

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The method of collocation with nonlinear programming is applied to the determination of minimum-time, low-thrust interplanetary transfer trajectories. Since the vehicle motor operates continuously, the minimum-time trajectories are also propellant minimizing. The numerical solution method requires that the transfer be divided into three phases: escape from the departure planet, heliocentric flight, and arrival at the destination planet. Two-body gravitational models are used in each phase and the transformation from planetocentric coordinates to heliocentric coordinates and vice-versa is incorporated as a set of nonlinear constraints on the problem variables. No a priori assumptions on the optimal control time history are required. An Earth-to-Mars transfer with a very low thrust acceleration of 0.0001 *g* is used as an example.

Introduction

THE subject of trajectory optimization for low-thrust vehicles has received a great deal of attention; there is a wealth of research papers on the subject of low-thrust interplanetary transfer alone, dating from the 1960s.[1–8] The reason why this field has been so popular, despite the fact that low-thrust propulsion has not yet been used for interplanetary flight, is the dramatic advantage electric propulsion offers in payload mass fraction compared to chemical propulsion.

The optimization of a low- (continuous-) thrust trajectory is qualitatively different from that of an impulsive thrust trajectory. The latter is essentially a finite parameter optimization problem where the times, directions, and magnitudes of the required impulses are to be determined and in which a state transition matrix is available to propogate the trajectory between impulses. A feasible trajectory may then be improved, perhaps by the addition of coast arcs or additional impulses, using, e.g., well-known methods associated with Lawden,[9] Lion and Handelsman,[10] or Jezewski and Rozendaal.[11]

The continuous-thrust problem is more difficult to solve because the trajectory cannot be propagated forward without numerical integration. Optimization theory applied to this continuous problem yields a two-point boundary value problem (TPBVP). Such TPBVPs are generally very difficult to solve. There are two basic approaches: indirect and direct. Indirect solutions for the optimal control explicitly use the necessary conditions of optimality (the Euler–Lagrange equations) and hence the system costate (or Lagrange multiplier) variables. Among the first and best known of the indirect solution methods are initial value, or "shooting," methods, in which one guesses either the initial or terminal boundary conditions, then integrates numerically forward or backward, and finally readjusts this initial guess iteratively so that the boundary conditions at the other end are also satisfied. These methods, however, are extremely sensitive to the accuracy of the initial guess, thus making the numerical computation for an improved guess quite difficult. The direct approach is to discretize the original problem and transform it into a parameter optimization problem. Explicit integration of the system differential equations is avoided. Instead, algebraic expressions (e.g., Hermite cubic polynomials) approximate the differential equations locally, and the resulting system of nonlinear simultaneous equations is then solved by mathematical programming.[12–17] This is the method of solution that will be used in this work; it will be described in more detail in a subsequent section.

Received Aug. 23, 1993; revision received Aug. 21, 1994; accepted for publication Oct. 1, 1994.

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The customary and reasonable assumption made in determining interplanetary trajectories is that they may be constructed by piecing together planetocentric and heliocentric trajectories that each use two-body gravitational models.[5,8,18] The inclusion of the planetocentric portions of the trajectory, which normally involve a many-revolution, spiral orbit, in the overall optimization is problematic. An exact description of the planetocentric segments requires the use of generalized coordinates appropriate to each planetary two-body problem. Thus three systems of coordinates are needed for the interplanetary transfer and each set will vary through a different range, necessitating scaling of the variables prior to the numerical optimization to prevent problems of sensitivity. Early studies (1960s) avoided this difficulty by simply neglecting the planetocentric portions of the trajectory.[1,2,4,6] Another approach is to make some a priori assumptions about the planetocentric spiral orbits, for example, assuming tangential thrusting.[5,8] The final states of the departure spiral orbit (or initial states for the arrival spiral) may then be determined in an approximate way analytically as a function of the flight time, the vehicle mass, and the performance characteristics of its motor. Thus only the flight times for the two planetocentric portions need appear as variables in the problem. The difficulty with this assumption, as will be shown here, is that the optimal thrust angle departs significantly from being tangential to the radius vector for much of the time of flight.

That the problem remains a difficult one to solve without these a priori assumptions about the optimal control is illustrated by a recent paper by Pierson and Kluever describing an optimal low-thrust Earth-Moon transfer[18]. Remarking that "the prospect of obtaining the minimum-fuel LEO-LLO trajectory by solving numerically a single optimization problem seems very remote" they proceed to solve the problem in three stages, each of which yields an extremal trajectory. The solution process in this work is qualitatively similar, in that three separate extremals are determined for the departure, heliocentric flight, and arrival trajectories. These solutions are then pieced together to provide an initial guess, which is in fact a feasible but suboptimal solution, for the nonlinear programming problem (NLP) solver. No a priori assumptions used in generating the initial guesses need be preserved by the NLP problem solver, except perhaps for specified initial and terminal state constraints, so that the final solution is indeed optimal. However, once one solution has been obtained it is usually not necessary to generate the three separate solutions to form an initial guess. If the "new" problem resembles one previously solved, but with modest changes, e.g. in initial or terminal states, or thrust level, the NLP problem solver can use the existing solution as the initial guess.

An optimal transfer trajectory from geosynchronous to areosynchronous (Martian-synchronous) orbit is determined as an example.

A vehicle thrust acceleration of 0.0001 g (0.00098 m/s^2) has been selected as representative of current continuous low-thrust spacecraft. For simplicity, circular and coplanar Earth and Martian orbits are assumed, and the spacecraft mass variation as fuel is consumed is ignored.

Governing Equations

The governing equations of motion (EOMs) are derived from Newton's law of gravitation and second law of motion. As previously described, the solution to this problem is divided into three phases: escape from the departure planet, heliocentric flight, and arrival at the target planet. We have used the patched conic approximation here; i.e., in each phase only one body attracts the spacecraft. The transition from planetocentric motion to heliocentric motion (and vice versa) is made at the sphere of influence of the planet. In polar coordinates (x, θ) the EOMs then become

$$\begin{aligned}
\dot{r} &= v_r \\
\dot{\theta} &= (v_\theta / r) \\
\dot{v}_r &= (v_\theta^2 / r) - (\mu / r^2) + a_T \sin\beta \\
\dot{v}_\theta &= -(v_r v_\theta / r) + a_T \cos\beta
\end{aligned} \tag{1}$$

where μ is the gravitational constant of the attracting body, β is the thrust pointing angle, and a_T is the thrust acceleration magnitude. Equations (1) describe motion of the spacecraft in a plane, in this case the ecliptic plane. We thus assume that the target planet also lies in the ecliptic plane, which is not a bad assumption for many of the planets to which spacecraft have been sent. For the example of an Earth–Mars transfer described later in this work the 1.85 deg inclination of the Martian orbit to the ecliptic may be expected to have a negligible effect on the cost of the transfer.[7] A perhaps less justifiable assumption, following from Eqs. (1), is that the initial orbit of the spacecraft about Earth and final orbit about the target planet also lie in the ecliptic plane. This restriction will be removed in future work but was considered reasonable here in our first attempt at generating an optimal interplanetary trajectory.

For reasons of sensitivity it is necessary that the EOMs be expressed in local variables for each of the three phases. Thus coordinate transformations are necessary at the boundary of two phases to provide initial conditions for the integration of Eqs. (1). For example, assuming that the spacecraft is departing Earth, transformation from geocentric polar coordinates (and corresponding velocities) to heliocentric polar coordinates is done when the spacecraft reaches Earth's sphere of influence. This is illustrated in Fig. 1. Transformation from the coordinates and velocities $(r, \theta, v_r, v_\theta)$ to $(R, \gamma, v_R, v_\gamma)$ may be accomplished using the equations[19]

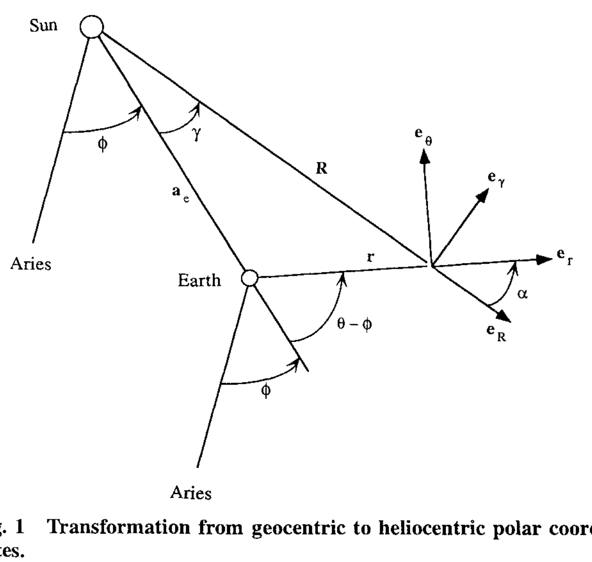

Fig. 1 Transformation from geocentric to heliocentric polar coordinates.

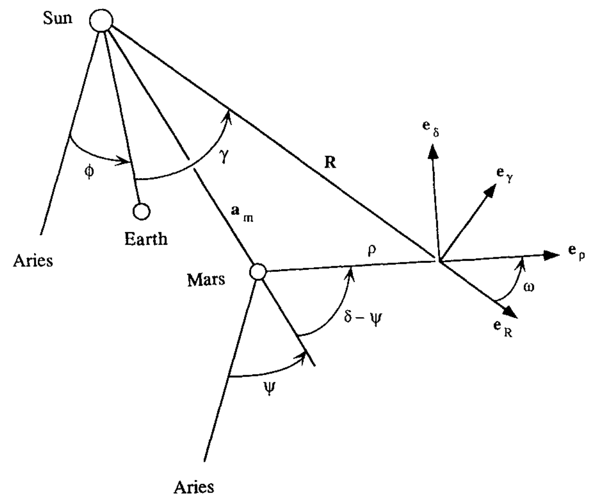

Fig. 2 Transformation from heliocentric to Mars-centered polar coordinates.

$$\begin{aligned}
R\cos\alpha &= a_E \cos(\theta - \phi) + r \\
R\sin\alpha &= a_E \sin(\theta - \phi) \\
v_\gamma \cos\alpha(1 + \tan^2\alpha) &= a_E \dot{\phi} \cos(\theta - \phi) + v_\theta \\
&\quad + a_E \dot{\phi} \sin(\theta - \phi)\tan\alpha + v_r \tan\alpha \\
v_R \cos\alpha &= a_E \dot{\phi} \sin(\theta - \phi) + v_r - v_\gamma \sin\alpha
\end{aligned} \tag{2}$$

where a a_E is the semimajor axis of Earth's orbit, $\alpha = \theta - \phi - \gamma$, and $\dot{\phi}$ is the angular velocity of Earth in orbit about the sun. Equations (2) may be solved algebraically for R and γ, but the resulting equations involve arctangents and prove to be less satisfactory as equations of constraint than the equations above, as will be described in the next section.

A similar transformation is required from the heliocentric polar coordinates and velocities to planetocentric polar coordinates and velocities when the spacecraft reaches the sphere of influence of the target planet. This is illustrated in Fig. 2 for the case in which the arrival planet is Mars. Transformation from the heliocentric coordinates and velocities $(R, \gamma, v_R, v_\gamma)$ to the corresponding planetocentric quantities $(\rho, \delta, v_p, v_\delta)$ is governed by the equations[19]

$$\begin{aligned}
0 &= R\sin\omega - a_m \sin(\delta - \psi) \\
\rho &= R\cos\omega - a_m \cos(\delta - \psi) \\
v_\delta &= v_\gamma \cos\omega - v_R \sin\omega - a_m \dot{\psi} \cos(\delta - \psi) \\
v_\rho &= v_R \cos\omega + v_\gamma \sin\omega - a_m \dot{\psi} \sin(\delta - \psi)
\end{aligned} \tag{3}$$

where a_m is the semimajor axis of the Martian orbit about the sun, $\omega = \delta - \phi - \gamma$, and $\dot{\psi}$ is the instantaneous angular velocity of Mars in its orbit about the sun.

After the first transformation of variables (as the spacecraft leaves Earth's sphere of influence) the system governing equations remain Eqs. (1) but with r replaced by R and θ replaced by γ. The gravitational constant must of course be replaced by that of the sun. Normalized variables are used in each phase; e.g., for the geocentric motion the unit of distance is chosen to be the Earth radius and the unit of time is such that 2π time units represent the period of an Earth orbit at Earth's surface. For the heliocentric motion the distance unit becomes 1 AU and 2π time units equal one Earth year. Similar normalized units are used for the planetocentric motion within the sphere of influence of the target planet. For each of the three phases of flight the gravitational constant μ will be unity expressed in terms of these normalized distance and time units.

The coordinate transformations (2, 3) have been simplified considerably by the assumption that the orbits about the sun of the planets of departure and destination are circular. For some solar system interplanetary transfers this assumption would be entirely reasonable. For the Earth-to-Mars trajectory used as an example in a later section this assumption is not especially good. The trajectory obtained may however be considered a compromise between the most favorable and least favorable transfers having the same initial angle by which Mars leads Earth at departure.[7] It would be straightforward to modify the coordinate transformations to accommodate elliptical planetary orbits.

Method of Solution

As described in the previous sections, the interplanetary trajectory is constructed from three two-body problems with coordinate transformations at the boundaries between planetocentric and heliocentric orbits. Initial conditions and terminal conditions are thus described using functions of different sets of generalized coordinates; the equations of motion change as the attracting center changes from one body to another. Necessary conditions for optimality for such a system with discontinuities in the state variables at interior points are given in Bryson and Ho.[20] An algorithm for the solution of such a problem using an indirect method such as "shooting" would face considerably more difficulty (than would already exist for a problem without such discontinuities) since the shooting would need to be done between each pair of neighboring boundary/interior points, from additional unspecified boundary/interior initial/final values of the costate or state variables. This approach was not considered likely to be successful.

The method of solution chosen here is a direct method utilizing collocation and nonlinear programming (DCNLP). It is described in a paper by Hargraves and Paris[14] and forms the basis of their program, OTIS, for the optimization of atmospheric vehicle trajectories. The method is based on a collocation scheme developed by Russell and Shampine[12] for solving boundary value problems using piecewise polynomials. Dickmanns and Well[13] also described a solution method using Hermite polynomials and collocation, but their work explicitly retains the Lagrange multipliers of the problem and requires explicit satisfaction of the boundary conditions on these multipliers as well as the system transversality condition; the principal contribution of the Hargraves and Paris[14] paper is to show that problems can be solved robustly without explicitly including the Lagrange multipliers. Conway and his students have had success applying the DCNLP method to the solution of a variety of optimal space trajectory problems.[15,17,22]

Since a thorough description of the DCNLP method is available elsewhere,[13,14,15,17] it will be described very succinctly here. However, techniques required to adapt this specific problem to the DCNLP structure, particularly those required for the piecing together of planetocentric and heliocentric trajectories, will be described completely. The optimal control problem is first discretized into a sequence of stages or events E_i. Given the vehicle parameters (e.g., constant thrust magnitude), the control history $\boldsymbol{u}(t)$ and the times that define the duration of important events or phases are chosen to minimize the cost function:

$$J = \phi[\boldsymbol{x}(\boldsymbol{E}), \boldsymbol{u}(\boldsymbol{E}), \boldsymbol{E}] \tag{4}$$

where $\boldsymbol{E}$ is a vector of event time variables, $\boldsymbol{x}$ is the vector of state variables, and $\boldsymbol{u}$ is the vector of control variables. Within each stage, the state variables must satisfy the system governing Eqs. (1), which may be compactly written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \tag{5}$$

In the direct collocation method applied in this work, the state time histories are discretized into N time segments, as shown in Fig. 3. The vector $\boldsymbol{x}$ consists of $n(N+1)$ parameters; the discrete values of all n state variables at $N+1$ segment boundaries (termed "nodes"). At each node, the current values of the discrete states and the values of the discrete controls from the vector $\boldsymbol{u}$ can be used to evaluate Eqs. (5); i.e., the slope of the function $x_i(t)$ may be determined for all n states at each segment boundary. Within each segment, knowledge of the states at each boundary and state derivatives at each boundary is used to determine unique cubic polynomials to span the segment. These piecewise cubics will approximate the state histories; the control histories are represented linearly.

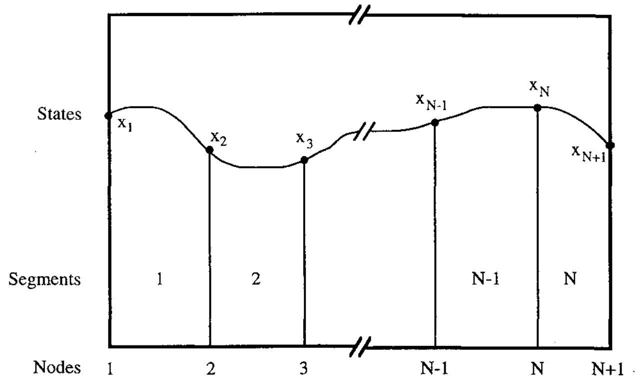

Fig. 3 Discretization of state variable histories.

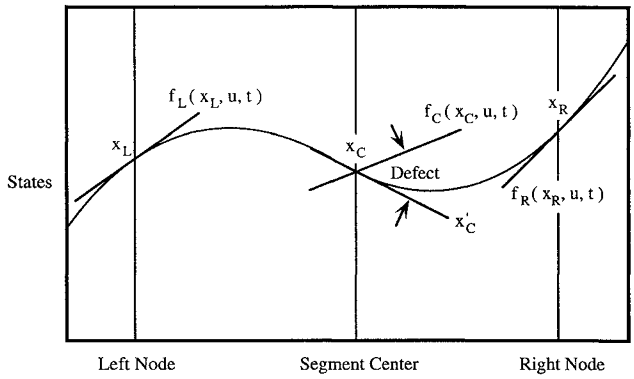

Fig. 4 Formation of nonlinear "defect" equations.

To satisfy the differential equations at the center of each segment, Hermite interpolation is used to evaluate the states $\boldsymbol{x}_c$ and their corresponding time derivatives $\dot{\boldsymbol{x}}_c$ at the segment midpoint, as shown in Fig. 4. The differences between the interpolated representation and that derived from the differential equations, $\dot{x}_C - f(x_C, u_C, t_C)$, for each state variable in each segment constitute a set of nonlinear "defect" equations. These defects become some of the nonlinear constraints of the nonlinear programming (NLP) problem; if they can be driven to zero as the NLP problem solver selects the discrete values of the state and control variables (i.e., $\boldsymbol{x}$ and $\boldsymbol{u}$), then each cubic polynomial is accurately representing the system differential Eqs. (1) at the left and right boundaries of the segment and its center.

Collecting all the independent variables into a single vector $\boldsymbol{P}$ defined as

$$\boldsymbol{P}^T = [\boldsymbol{Z}^T, \boldsymbol{E}^T] \tag{6}$$

where

$$\boldsymbol{Z}^T = \left(\boldsymbol{x}_1^T, \boldsymbol{u}_1^T, \boldsymbol{x}_2^T, \boldsymbol{u}_2^T, \ldots, \boldsymbol{x}_{N+1}^T, \boldsymbol{u}_{N+1}^T\right)$$

and similarly collecting all the nonlinear constraints into the vector $\mathbf{C}^T$, an optimal control problem can then be restated as a NLP problem of the form:

Minimize

$$\phi(\boldsymbol{P})$$

Subject to

$$\boldsymbol{b}_L \le \begin{Bmatrix} \boldsymbol{P} \\ \boldsymbol{AP} \\ \mathbf{C}(\boldsymbol{P}) \end{Bmatrix} \le \boldsymbol{b}_U$$

where $\boldsymbol{AP}$ contains all the linear constraints of the stated problem and $\boldsymbol{b}_L$ and $\boldsymbol{b}_U$ are the lower and upper bounds on the variables and constraints. The linear and nonlinear constraints have upper and lower bounds that are both zero, i.e., they are represented by equalities.

The NLP code used for this work is NZSOL, an improved version of the NPSOL program used by Hargraves and Paris.[14,23] Subroutines coded in FORTRAN-77 were also used to evaluate the objective function and its gradient as well as the constraints and their Jacobian, which were then supplied to the NZSOL program along with an initial guess via the main program. These programs were then executed on a Convex C240 computer.

In this work, time segments of equal length are used within each of the transfer phases. For efficient representation, more segments are devoted to the Earth departure phase (EDP) and the Mars arrival phase (MAP) than to the heliocentric phase (HP) of flight. In both the departure and arrival phases, the long spiral trajectories required a large number of cubic polynomials to accurately represent the states. In contrast, fewer segments are necessary in the HP because, when expressed in HP units, the variations in the states are relatively small.

In addition, for each of the coordinate transformations, a time segment of zero length is defined at the interface of the adjacent phases. Then states at the left node represent the final state values of the previous phase whereas those at the right represent the initial state values of the next phase. The states at the left node are then converted to those at the right via the transformation equations, e.g., Eqs. (3) for the EDP-to-HP coordinate transformation. These equations relating the states at the interface are nonlinear. They are thus included in the problem [through the vector $\boldsymbol{C}(\boldsymbol{P})$] as additional nonlinear constraints.

The requirement of performing the coordinate transformations at the respective planets' sphere of influence is satisfied by fixing the state variables corresponding to the final distance from Earth in the departure phase (r) and the initial distance from Mars in the arrival phase (ρ) to be the respective sphere-of-influence distances. Initial and terminal orbits and orbital velocities are also specified in this manner, e.g., in this problem the initial orbit is geosynchronous and the terminal orbit is Martian synchronous.

In the largest problem solved, the EDP and MAP are each divided into 80 segments whereas the HP is divided into 20 segments. The total number of variables is 918 [183 nodes $\times$ (4 states+1 control)+ $t_1 + t_2 + t_3$], the t_i being the durations of each of the three phases of flight, (EDP, HP, and MAP), and the total number of constraints is 728 (720 defects + 4 for the ECT + 4 for the MCT). The execution time for this routine is about 5 h on the Convex C240.

An initial guess of the solution (for all the state and control variables at all the discrete times) must be provided to the NLP problem solver NZSOL. Due to the size of the complete EDP + HP + MAP problem, attempts at starting these problems with linear guesses have been unsuccessful. Therefore, simpler problems such as individual EDP, HP, or MAP problems are first solved using linear guesses and physically reasonable objective functions. The objective function in the EDP and HP problems is the final orbit energy, and the transfer time is fixed. In the MAP problem, transfer time is open and the objective is to minimize this transfer time. Separate programs are used to perform the ECT and MCT. Thus the final states of one phase are used as the initial states for the next phase. The solutions from these three single-phase problems are then pieced together and used as the initial guess for the complete problem. Note that these separate solutions together constitute a feasible, but not minimum-time, solution for the complete problem. The single-phase EDP and MAP solutions each using 80 segments required about 1 h of execution time whereas the HP solution of 20 segments took only about 10 min, in comparison to the 5 h mentioned above as being required for the solution of the complete (182-segment) problem.

As the size of the problem increases, the likelihood of failure to converge to an optimal solution also increases. For large problems, NZSOL would occasionally fail to converge even when a feasible solution was given as its initial guess. Therefore, whenever possible, the routine was first executed with a reasonably small but adequate number of segments. Then the solution from this run was interpolated and provided as the initial guess for a more accurate run using more segments.

Results

As an example, the minimum-time transfer from geosynchronous orbit (a circular orbit at approximately 6.6 Earth radii) to areosynchronous orbit (circular orbit at approximately 6.0 Martian radii) was obtained assuming a constant thrust acceleration of 0.0001 g (9.8×10^{-4} m/s^2) and assuming a launch date on which Mars leads Earth by 0.9666 rad. Figure 5 shows the trajectory of the EDP near Earth. The thrust angle during the EDP, measured with respect to the local horizontal, i.e., the normal to the radius vector, is shown in Fig. 6. The spacecraft reaches Earth's sphere of influence after 33.08 days. Note that for the latter half of this 33-day escape the thrust angle becomes quite large, so that the assumption of tangential thrust used by earlier researchers[5,8] is not accurate. The eccentricity of the Earth departure orbit is shown in Fig. 7. The orbit has become hyperbolic, i.e., the spacecraft has escaped Earth, after approximately 28 days. The HP of flight and the initial positions of Earth and Mars are shown in Fig. 8. During the HP, lasting 169.78 days, the thrust angle changes dramatically, as shown in Fig. 9. For much of the latter part of the HP the thrust is used to reduce the heliocentric velocity. This is to allow the spacecraft to be captured into orbit about Mars despite having a motor capable of only a very small thrust acceleration. The eccentricity of the heliocentric orbit, shown in Fig. 10, becomes very small as the spacecraft arrives at Mars for the same reason, to yield only a small difference between the absolute velocities of the spacecraft

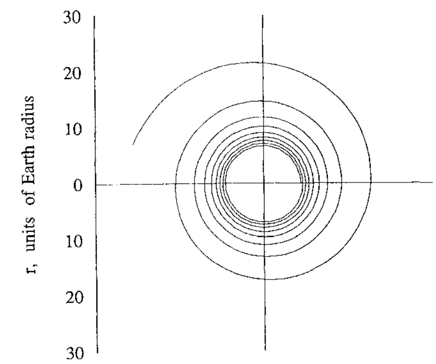

Fig. 5 Earth departure trajectory.

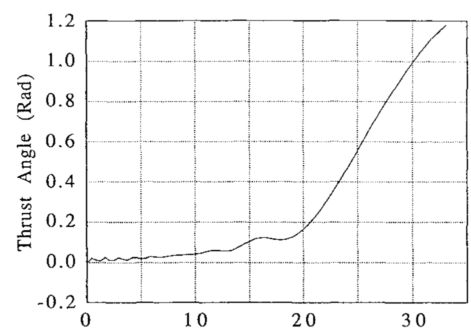

Fig. 6 Thrust angle history during Earth departure.

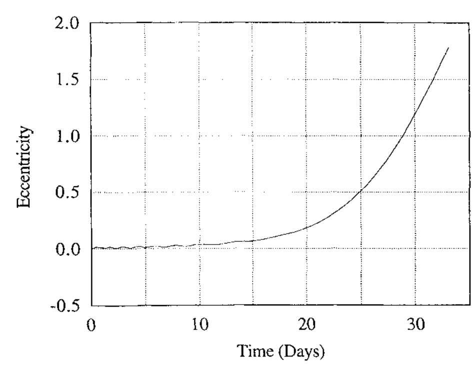

Fig. 7 Orbit eccentricity during Earth departure.

Table 1 Total transfer time as a function thrust acceleration magnitude

Acceleration, milli-g	Flight time, days
0.098	229.15
0.099	224.37
0.100	222.14
0.105	218.18
0.110	214.51

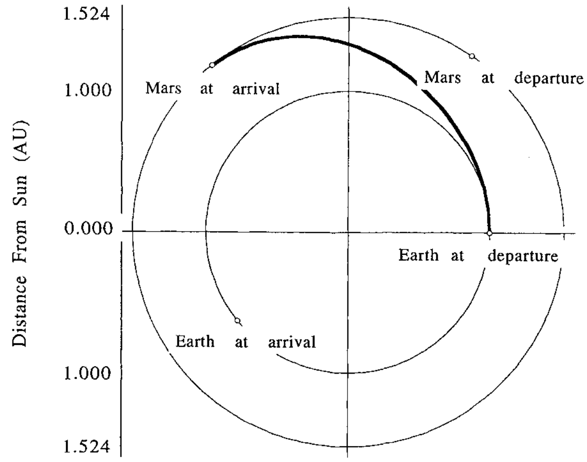

Fig. 8 Heliocentric trajectory.

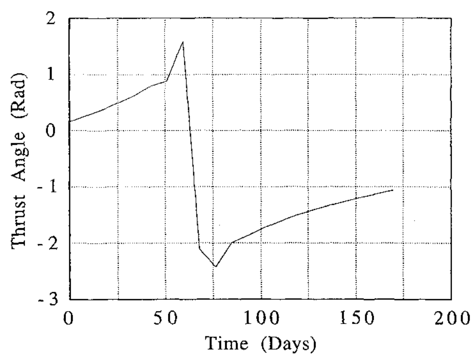

Fig. 9 Thrust angle history during heliocentric flight.

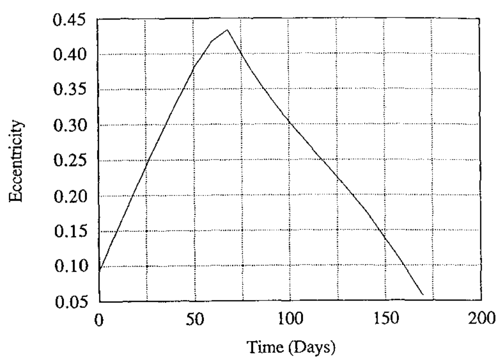

Fig. 10 Orbit eccentricity during heliocentric flight.

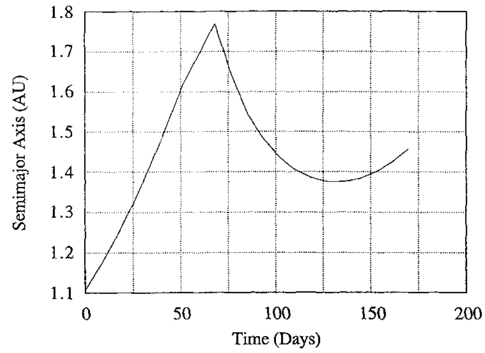

Fig. 11 Semimajor axis history during heliocentric flight.

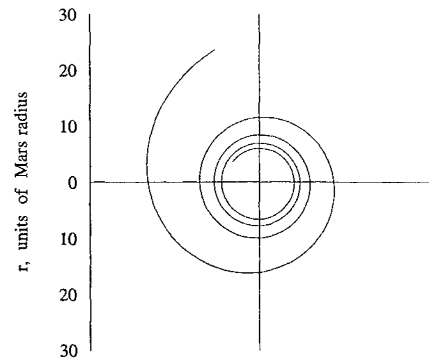

Fig. 12 Mars arrival trajectory.

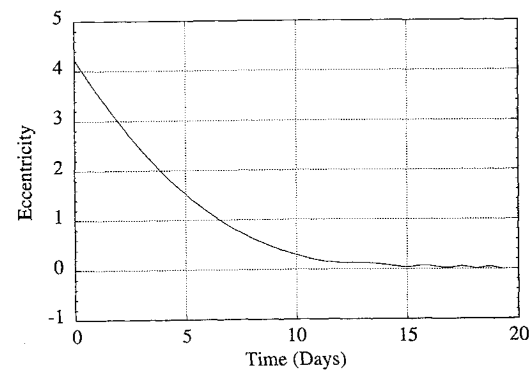

Fig. 13 Orbit eccentricity during Mars arrival.

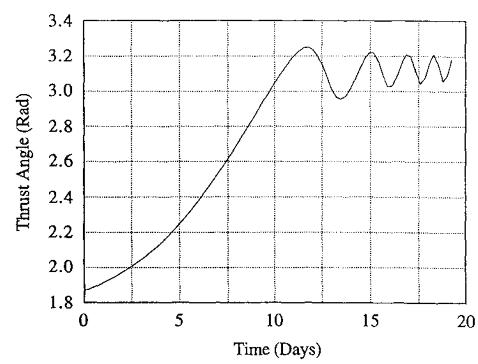

Fig. 14 Thrust angle history during Mars arrival.

and Mars. The semimajor axis of the heliocentric orbit is shown in Fig. 11.

The MAP trajectory near Mars is shown in Fig. 12. From arrival at the Martian sphere of influence to areosynchronous orbit 19.28 days are required, for a total minimum flight time of 222.14 days. For approximately the first 7 of the 19.28 days of the MAP the spacecraft is still on a hyperbolic trajectory with respect to Mars, as shown in Fig. 13. The thrust angle during the MAP is essentially retrograde, as shown in Fig. 14.

The minimum-flight-time trajectories have been obtained for a number of different launch dates, i.e., for a number of different angles by which Mars leads Earth at launch. The result is shown in Fig. 15; there is a significant variation in flight time, as expected. The minimum (217.6 days) over all of the launch geometries considered is likely to be only a local minimum since such a small range of initial angular separations of the two planets was considered.

The original example described above in which Mars leads Earth at launch by 0.9666 rad was run with a number of different thrust

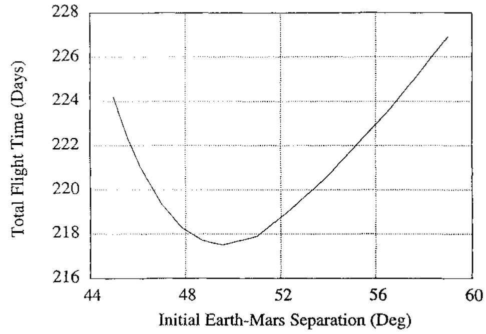

Fig. 15 Total minimum flight time as a function of Earth–Mars angular separation at launch.

acceleration magnitudes. Total minimum transfer time as a function of thrust acceleration is shown in Table 1.

Unlike many other optimal control problems solved using this method, the solution here is unfortunately not very robust. This prevented us from including the launch date (i.e., launch geometry) as a variable to be optimized.

The lack of robustness is probably due to the fact that this is a very ambitious problem considering the capability of the NLP solver. Our example Earth–Mars transfer had 918 NLP variables and 728 constraints, all but a few of which were nonlinear equations. A "large" problem, as defined by the developer of the NLP problem-solving program NZSOL, is one of 800 variables, so the problem solved here is at the boundary of feasibility. An additional problem is that even with normalization of variables in each of the three phases into which the problem is divided the variables change over a large range. This is most acute for the departure and arrival trajectories.

These problems may be substantially alleviated by changing the type of "defect" used as a NLP problem constraint to ensure the satisfaction of the system differential equations. Enright and Conway[17] and Scheel and Conway[21,22] have shown that the size of the problem may be significantly reduced by using parallel shooting and Runge–Kutta defects. The advantage of this approach is that it retains the original discretization of the control variables, so that control is kept reasonably continuous, but effectively uses a much coarser discretization for the state variables, so that there are many fewer of them. Since there are typically many more state variables than control variables in a given problem, e.g., in this problem there are four states and one control, the size of the NLP problem can be significantly reduced.

Another possible approach to reducing the size of the NLP problem while retaining the same solution accuracy is based on the work of Herman and Conway.[24] In this work Jacobi polynomials of higher degree than the third-degree Hermite polynomials used in the Hargraves–Paris collocation method[14] are used to represent the state variable histories in each "segment," implicitly yielding integration rules of higher order and thus smaller integration error. The number of segments used and thus the number of discrete state variables used may be significantly reduced without decreasing the solution accuracy. It is likely that one or the other of these two approaches will be required to solve this orbit transfer problem for the noncoplanar case, which requires significantly more state variables and an additional (thrust pointing angle) control variable. This will be the subject of future research.

Conclusions

This work demonstrates that the method of direct collocation with nonlinear programming may be successfully applied to the problem of optimization of very low thrust interplanetary transfers, including the departure and arrival phases of flight, with no a priori assumptions regarding the form of the optimal control history. The problem has been simplified by assuming that the spacecraft is always in the ecliptic plane, that the planetary orbits about the sun are circular, and that the spacecraft mass does not change. However, all three of these simplifying assumptions could be removed without significant change to the way in which the problem is solved. Using the collocation solution method, it is straightforward to change the objective function so that the method described here will apply equally well to finding minimum-fuel trajectories for power-limited vehicles as opposed to the constant-thrust vehicle used in this example Earth–Mars trajectory. The example trajectory shows that a commonly used simplifying assumption, of tangential thrust during the planetary departure and arrival orbits, is not valid.

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